HOMOLOGY: THEORETICAL AND COMPUTATIONAL ASPECTS

Massimo Ferri, Patrizio Frosini, Claudia Landi

Generalized persistent homologies:
G-invariant and multi-dimensional persistence

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Exercises on G-invariant persistence

(1) Let Φ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$, endowed with the sup-norm. Let S be a subset of the group $\mathrm{Homeo}([0,1])$ of the self-homeomorphisms of $[0,1]$. Is

$$d_S(\varphi_1, \varphi_2) := \inf_{g \in S} \|\varphi_1 - \varphi_2 \circ g\|_\infty$$

a pseudo-metric on Φ? Justify the answer.

(2) Let Φ be the set of all continuous functions from $[0,1]$ to $\mathbb{R}$, endowed with the sup-norm. Find a subgroup G of the group $\mathrm{Homeo}([0,1])$ of the self-homeomorphisms of $[0,1]$ and two functions $\varphi_1, \varphi_2 \in \Phi$, such that

$$d_G(\varphi_1, \varphi_2) := \inf_{g \in G} \|\varphi_1 - \varphi_2 \circ g\|_\infty < \|\varphi_1 - \varphi_2 \circ \bar{g}\|_\infty$$

for every $\bar{g} \in G$. Can G be compact?

(3) Let Φ be the set of all continuous functions from X to $\mathbb{R}$, endowed with the sup-norm. Let G be a subgroup of the group $\mathrm{Homeo}(X)$ of the self-homeomorphisms of X. If H is a subgroup of $\mathrm{Homeo}(X)$ such that

(a) H is finite (i.e. $H = \{h_1, \ldots, h_r\}$);
(b) $g \circ h \circ g^{-1} \in H$ for every $g \in G$ and every $h \in H$;

we say that H is *associated with* G.

i): Find a group H associated with G in the case that
- $X = \mathbb{R}^2$ and G is the group of the "horizontal" translations of $\mathbb{R}^2$;
- $X = S^1$ and G is the group of the rotations of S^1;
- $X = \mathbb{R}^2$ and G is the group of the isometries of $\mathbb{R}^2$.

ii): Is it possible to find a group H associated with G for any subgroup G of $\mathrm{Homeo}(S^1)$?

(4) Find a set Φ of continuous functions from $[0,1]$ to $\mathbb{R}$ (endowed with the sup-norm) and a subgroup G of $\mathrm{Homeo}([0,1])$ such that the set of G-invariant non-expansive operators from Φ to Φ is not compact.

(5) G-invariant non-expansive operators can be defined also when the space X is not compact. Find G-invariant non-expansive operators in the case that
- $X = \mathbb{R}^2$, $\Phi = C^0(\mathbb{R}^2, \mathbb{R})$ and G is the group of translations;
- $X = \mathbb{R}^2$, $\Phi = C^0(\mathbb{R}^2, \mathbb{R})$ and G is the group of isometries.

The slides of the lecture about G-invariant persistence are available at this link:
`http://www.dm.unibo.it/~frosini/pdf/Lecture_Frosini_HTCA_2015.pdf`