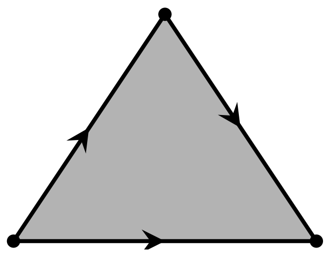
HTCA15 - GENOVA
9-13 FEBRUARY 2015
COURSE ON DISCRETE MORSE THEORY

First assignment

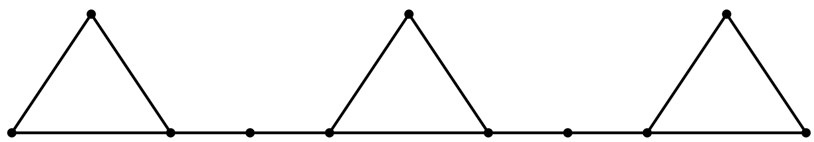
(1)
- Prove that every graph can be drawn in R^3 without intersections.
- Prove that every d-dimensional abstract simplicial complex can be realized in $\mathbb{R}^{2d+1}$ as geometric simplicial complex.

(2) Let D be the Dunce Hat (the 2-dimensional space obtained by identifying the 3 edges of a triangle as in the figure).



- Find a triangulation T of D. [*Hint:* you need to add few extra points.]
- The Dunce Hat is contractible. Show that T is not collapsible.

(3) Let G be the graph in the picture:



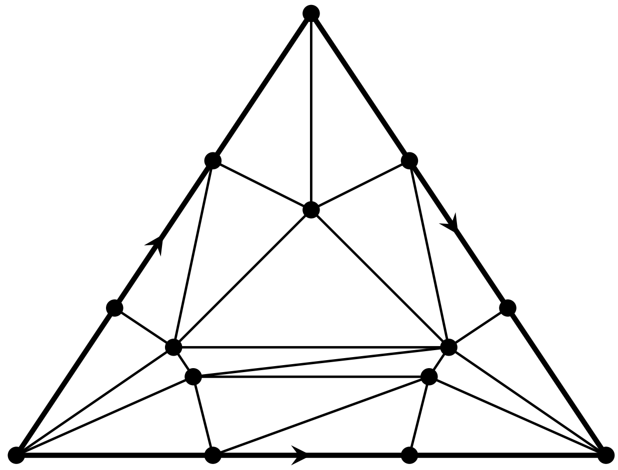
- Is G collapsible? Why?
- Find the minimal number k of edges we have to delete in order to make G contractible. Is any choice of k edges equivalent?
- Which 1-dimensional simplicial complexes are collapsible?
- Prove that a collapsible complex collapses onto any of its vertices.

(4) Prove that every triangulated disc is collapsible.

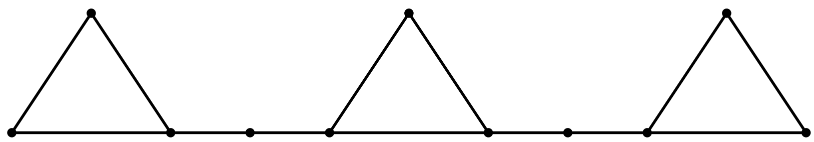
Second assignment

(1) Let f be a discrete Morse function.
- Prove that if σ and τ are two faces such that $f(\sigma) = f(\tau)$, then $|\dim(\sigma) - \dim(\tau)| = 1$.
- Prove that f has at least one critical vertex.
- Find a complex that has no critical facet.

(2) Compute all possible Morse vectors of the following triangulation of the Dunce Hat:



(3) Look again at the following graph:



- How many different Morse vectors can we get if we run the random algorithm of Benedetti-Lutz?
- If we run the algorithm once, what is the probability of getting the optimal Morse vector?
- Can you find examples that make such probability arbitrarily small?

(4) Let G be a tree (that is, a connected acyclic graph).
- Prove that G has at least 2 leaves.
- Prove that G embeds in $\mathbb{R}^2$.

Third assignment

(1) Let C and C' two collapsible d-complexes.
  - If we identify a vertex v of C and a vertex v' of C', is the new complex collapsible?
  - What if we identify two $(d-1)$-faces?

(2) Suppose that a complex C admits a discrete Morse function with c_i critical i-faces. Prove that also its barycentric subdivision $\mathrm{sd}(C)$ admits a discrete Morse function with c_i critical i-faces.

(3) Let H be the Hasse diagram of a complex C. The *Hasse diagram* is a graph whose vertices correspond to faces of C. Two vertices of H associated to faces σ and τ are connected by an edge if and only if $\sigma \subset \tau$ or $\tau \subset \sigma$ and $|\dim(\sigma) - \dim(\tau)| = 1$. A *matching* of H is a subset of the edges of H such that no two edges intersect in a vertex. We use a matching to induce an orientation on H in the following way: we orient all the edges in the matching from the face of smaller dimension to the one of higher dimension and all other edges from the higher dimensional face to the lower dimensional one. A matching of H is called *acyclic* if it induces an orientation of H such that the if we consider H as an oriented graph, it does not have directed cycles. Prove that:
  - for a discrete Morse function f on C,
  $$M(f) := \{\sigma \to \tau \mid \sigma \text{ is a facet of } \tau, f(\sigma) = f(\tau)\}$$
  is an acyclic matching of H.
  - for any acyclic matching M on H, there exists a discrete Morse function f on C such that $M = M(f)$.