Effective Homology: Perturbation Lemma and Applications

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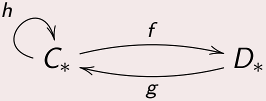
Introduction

- The Basic Perturbation Lemma was discovered by Shih Weishu in 1960, and the abstract modern form was given by Ronnie Brown in 1964 (based on unpublished results by Barrat).
- We have used it combined with the effective homology method, in order to determine:
  - Homology of cones, bicomplexes, twisted Cartesian products, loop spaces, classifying spaces...
  - Homotopy groups of spaces by means of Whitehead and Postnikov towers.
  - Homology of digital images by means of Discrete Vector Fields.
  - Spectral sequences associated with filtered complexes (including Serre and Eilenberg-Moore spectral sequences).
  - Persistent homology.
  - Koszul homology.
  - Bousfield-Kan spectral sequence for computing homotopy groups of spaces.
  - Homology of groups.
  - Neuronal images processing.

Effective homology

Definition

A *reduction* ρ between two chain complexes C_* and D_* (denoted by $\rho : C_* \Rrightarrow D_*$) is a triple $\rho = (f, g, h)$



satisfying the following relations:

1) $fg = \mathrm{Id}_{D_*}$;
2) $d_C h + h d_C = \mathrm{Id}_{C_*} - gf$;
3) $fh = 0; \quad hg = 0; \quad hh = 0.$

If $C_* \Rrightarrow D_*$, then $C_* \cong D_* \oplus A_*$, with A_* acyclic, which implies that $H_n(C_*) \cong H_n(D_*)$ for all n.

Definition

A *(strong chain) equivalence* ε between C_* and D_*, $\varepsilon : C_* \Lleftarrow\!\!\Rrightarrow D_*$, is a triple $\varepsilon = (B_*, \rho, \rho')$ where B_* is a chain complex, $\rho : B_* \Rrightarrow C_*$ and $\rho' : B_* \Rrightarrow D_*$.

Definition

An *object with effective homology* is a quadruple $(X, C_*(X), EC_*, \varepsilon)$ where EC_* is an effective chain complex and $\varepsilon : C_*(X) \Lleftarrow\!\!\Rrightarrow EC_*$.

This implies that $H_n(X) \cong H_n(EC_*)$ for all n.

Meta-theorem

Let $X_1, \ldots, X_k$ be a collection of objects with effective homology and Φ be a reasonable construction process:

$$\Phi : (X_1, \ldots, X_k) \to X.$$

Then there exists a version with effective homology Φ^{EH}

$$\Phi^{EH} : ((X_1, C(X_1), EC_1, \varepsilon_1), \ldots, (X_k, C(X_k), EC_k, \varepsilon_k)) \to (X, C(X), EC, \varepsilon)$$

The process is perfectly stable and can be again used with X for further calculations.

Examples: twisted Cartesian products, loop spaces, suspensions, simplicial Abelian groups generated by simplicial sets, ...

The Kenzo system

The Kenzo system uses the notion of *object with effective homology* to compute homology groups of some complicated spaces.

- If the complex is effective, then its homology groups can be determined by means of diagonalization algorithms on matrices.
- Otherwise, the program uses the effective homology.

Example:

$$X = \Omega(\Omega(\Omega(P^\infty\mathbb{R}/P^3\mathbb{R}) \cup_4 D^4) \cup_2 D^2)$$

$$H_5(X) = \mathbb{Z}_2^{23} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{16}$$

$$H_6(X) = \mathbb{Z}_2^{52} \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}^3$$

$$H_7(X) = \mathbb{Z}_2^{113} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8^3 \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{32} \oplus \mathbb{Z}$$

Perturbation theorems

Definition

Let (C_*, d) be a chain complex. A *perturbation* $\delta : C_* \to C_{*-1}$ is an operator of degree -1 satisfying $(d + \delta) \circ (d + \delta) = 0$.

This produces a new *perturbed* chain complex $(C_*, d + \delta)$

Let $\rho = (f, g, h)$ be a reduction

$$h \circlearrowright (C_*, d_C) \underset{g}{\overset{f}{\rightleftarrows}} (D_*, d_D)$$

What happens if we perturb d_C or d_D?

Perturbation theorems

Theorem (Trivial Perturbation Lemma, TPL)

Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ be a reduction, and δ_D a perturbation of d_D.

Then we have a new reduction:

$$h \circlearrowright (C_*, d_C + \delta_C) \underset{g}{\overset{f}{\rightleftarrows}} (D_*, d_D + \delta_D)$$

where $\delta_C = g \circ \delta_D \circ f$.

Theorem (Basic Perturbation Lemma, BPL)

Let $\rho = (f, g, h) : C_* \Rrightarrow D_*$ *be a reduction, and* δ_C *a perturbation of* d_C *such that the composition* $h \circ \delta_C$ *is pointwise nilpotent. Then we have a new reduction:*

$$h' \circlearrowright (C_*, d_C + \delta_C) \underset{g'}{\overset{f'}{\rightleftarrows}} (D_*, d_D + \delta_D)$$

where

- $\delta_D = f \circ \delta_C \circ \phi \circ g = f \circ \psi \circ \delta_C \circ g$;
- $f' = f \circ \psi = f \circ (\mathrm{Id}_{C_*} - \delta_C \circ \phi \circ h)$;
- $g' = \phi \circ g$;
- $h' = \phi \circ h = h \circ \psi$;

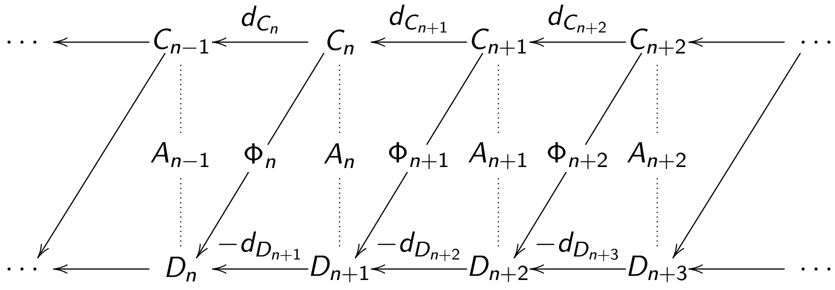
with the operators ϕ *and* ψ *defined by*

$$\phi = \sum_{i=0}^{\infty} (-1)^i (h \circ \delta_C)^i, \qquad \psi = \sum_{i=0}^{\infty} (-1)^i (\delta_C \circ h)^i = \mathrm{Id}_{C_*} - \delta_C \circ \phi \circ h$$

Algebraic cone construction

Definition

Let $\Phi : (C_*, d_C) \to (D_*, d_D)$ be a chain complex morphism. The *Cone* of Φ, $\text{Cone}(\Phi)_* \equiv (A_*, d_A)$, is a chain complex given by $A_n = C_n \oplus D_{n+1}$, with differential map $d_A(c, d) = (d_C(c), \Phi(c) - d_D(d))$.



Algebraic cone construction

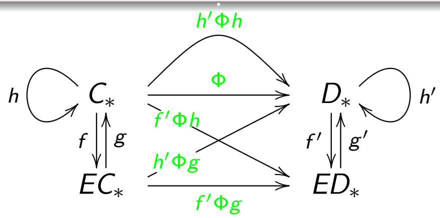
Theorem

A general algorithm can be produced:

- *Input:* $\Phi : C_* \rightarrow D_*$ *and effective homologies for* C_* *and* D_*.
- *Output: An effective homology for* $A_* = \text{Cone}(\Phi)$.

Proof:

1. Particular case $\Phi = 0$ (direct sum).
2. We install Φ.
3. We apply the BPL.

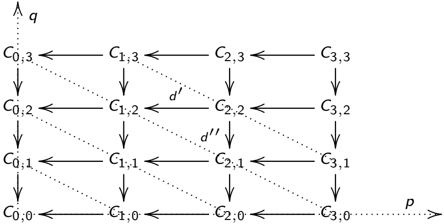


$$\underbrace{\begin{bmatrix} d_C & 0 \\ \Phi & -d_D \end{bmatrix}}_{D_A} \underbrace{\begin{bmatrix} d_{EC} & 0 \\ f'\Phi g & -d_{ED} \end{bmatrix}}_{D_{A'}} \underbrace{\begin{bmatrix} f & 0 \\ f'\Phi h & f' \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} g & 0 \\ -h'\Phi g & g' \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} h & 0 \\ h'\Phi h & -h' \end{bmatrix}}_{H}$$

Bicomplex

Definition

A *bicomplex* $C_{*,*}$ is a bigraded free $\mathbb{Z}$-module $C_{*,*} = \{C_{p,q}\}_{p,q\in\mathbb{Z}}$ provided with morphisms $d'_{p,q} : C_{p,q} \to C_{p-1,q}$ and $d''_{p,q} : C_{p,q} \to C_{p,q-1}$ satisfying $d'_{p-1,q} \circ d'_{p,q} = 0$, $d''_{p,q-1} \circ d''_{p,q} = 0$, and $d'_{p,q-1} \circ d''_{p,q} + d''_{p-1,q} \circ d'_{p,q} = 0$. The *total (chain) complex* $T_* = T_*(C_{*,*}) = (T_n, d_n)_{n\in\mathbb{Z}}$ is the chain complex given by $T_n = \bigoplus_{p+q=n} C_{p,q}$ and differential map $d_n(x) = d'_{p,q}(x) + d''_{p,q}(x)$ for $x \in C_{p,q}$.



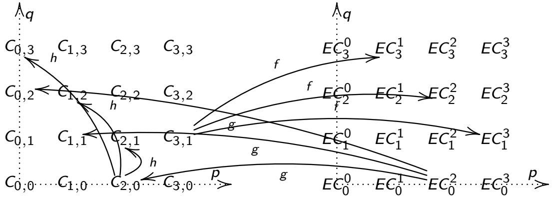
Theorem

A general algorithm can be produced:

- *Input: A bounded bicomplex* C_* *and effective homologies of each column.*
- *Output: An effective homology for* C_**.*

Proof:

1. We consider only the vertical arrows.
2. We *perturb* by adding the horizontal maps.
3. We apply the BPL.



Twisted Eilenberg-Zilber Theorem

Theorem (Eilenberg-Zilber)

Given two simplicial sets G and B, there exists a reduction

$$\rho = (f, g, h) : C_*(G \times B) \Rrightarrow C_*(G) \otimes C_*(B)$$

Theorem (Twisted Eilenberg-Zilber)

Given two simplicial sets G and B and a twisting operator $\tau : B \to G$, it is possible to construct a reduction

$$\rho = (f, g, h) : C_*(G \times_\tau B) \Rrightarrow C_*(G) \otimes_t C_*(B)$$

where $C_*(G) \otimes_t C_*(B)$ is a chain complex with the same underlying graded module as the tensor product $C_*(G) \otimes C_*(B)$, but the differential is modified to take account of the twisting operator τ.

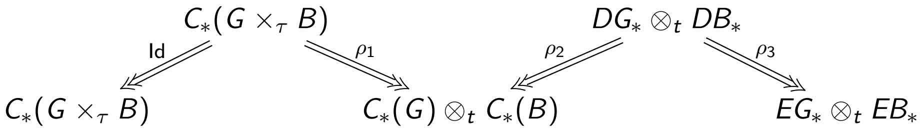
Proof: BPL.

Theorem

A general algorithm can be produced:

- *Input: two simplicial sets G and B (where B is 1-reduced), a twisting operator $\tau : B \to G$, and effective homologies for G and B.*
- *Output: An effective homology for $E = G \times_\tau B$.*

Proof: It is constructed as the composition of two equivalences:



where ρ_2 and ρ_3 are obtained by applying the TPL and the BPL respectively.

Theorem

A general algorithm can be produced:

- *Input: two simplicial sets G and B (where B is 1-reduced) and a twisting operator $\tau : B \to G$, and effective homologies for B and $E = G \times_\tau B$.*
- *Output: An effective homology for G.*

Proof: It is constructed as the composition of two equivalences:

$$C_*(G) \Lleftarrow \mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z}) \overset{\mathrm{Id}}{\Rrightarrow} \mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z})$$

$$\mathrm{Cobar}^{C_*(B)}(C_*(G) \otimes_t C_*(B), \mathbb{Z}) \Lleftarrow \widetilde{\mathrm{Cobar}}^{DB_*}(DE_*, \mathbb{Z}) \Rrightarrow \widetilde{\mathrm{Cobar}}^{EB_*}(EE_*, \mathbb{Z})$$

In particular, it can be applied for computing the effective homology of a loop space $\Omega(X)$, which is the fiber of a fibration $\Omega(X) \hookrightarrow \Omega(X) \times_\tau X \to X$ where the total space $E = \Omega(X) \times_\tau X$ is contractible, such that a reduction $C_*(\Omega(X) \times_\tau X) \Rrightarrow \mathbb{Z}$ can be built.

Discrete Morse theory

Definition

Let $C_* = (C_p, d_p)_{p \in \mathbb{Z}}$ a free chain complex with distinguished $\mathbb{Z}$-basis $\beta_p \subset C_p$. A *discrete vector field* V on C_* is a collection of pairs $V = \{(\sigma_i; \tau_i)\}_{i \in I}$ satisfying the conditions:

- Every σ_i is some element of β_p, in which case $\tau_i \in \beta_{p+1}$. The degree p depends on i and in general is not constant.
- Every component σ_i is a *regular face* of the corresponding τ_i.
- Each generator (*cell*) of C_* appears at most one time in V.

Definition

A V-path of degree p and length m is a sequence $\pi = ((\sigma_{i_k}, \tau_{i_k}))_{0 \leq k < m}$ satisfying:

- Every pair $((\sigma_{i_k}, \tau_{i_k}))$ is a component of V and τ_{i_k} is a p-cell.
- For every $0 < k < m$, the component σ_{i_k} is a face of $\tau_{i_{k-1}}$, non necessarily regular, but different from $\sigma_{i_{k-1}}$.

Discrete Morse theory

Definition

A discrete vector field V is *admissible* if for every $p \in \mathbb{Z}$, a function $\lambda_p : \beta_p \to \mathbb{N}$ is provided satisfying the following property: every V-path starting from $\sigma \in \beta_p$ has a length bounded by $\lambda_p(\sigma)$.

Definition

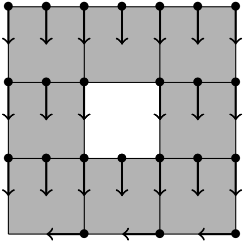
A cell σ which does not appear in a discrete vector field V is called a *critical cell*.

Theorem

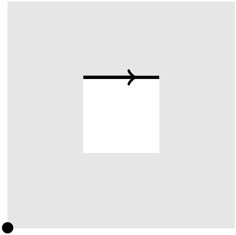
Let $C_* = (C_p, d_p)_{p\in\mathbb{Z}}$ be a free chain complex and $V = \{(\sigma_i; \tau_i)\}_{i\in I}$ be an admissible discrete vector field on C_*. Then the vector field V defines a canonical reduction $\rho = (f, g, h) : (C_p, d_p) \Rrightarrow (C_p^c, d_p')$ where $C_p^c = \mathbb{Z}[\beta_p^c]$ is the free $\mathbb{Z}$-module generated by the critical p-cells.

Proof: Uses BPL.

Discrete Morse theory and digital images



16 vertices
24 edges
8 squares



1 vertex
1 edge

Other applications of BPL and effective homology

- Homotopy groups of spaces by means of Whitehead and Postnikov towers.
- Spectral sequences of filtered complexes.
- Persistent homology.
- Koszul homology.
- Bousfield-Kan spectral sequence.
- Homology of groups.
- Neuronal images processing.

Homology of groups

Definition

A *resolution* F_* for a group G is an acyclic chain complex of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} F_{-1} = \mathbb{Z} \longrightarrow 0$$

A chain complex of Abelian groups is obtained: $\mathbb{Z} \otimes_{\mathbb{Z}G} F_*$

Theorem

Let G be a group and F_*, F'_* two free resolutions of G. Then

$$H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*) \cong H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F'_*) \cong H_n(K(G,1)) \quad \textit{for all } n \in \mathbb{N}$$

Definition

Given a group G, the *homology groups* $H_n(G)$ are defined as $H_n(G) = H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} F_*)$, $n \in \mathbb{N}$, where F_* is any free resolution for G.

One can always consider the *bar resolution* $B_* = \mathrm{Bar}_*(G)$, which satisfies $\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \equiv C_*(K(G,1))$. Drawback: for $n > 1$, $K(G,1)_n = G^n$.

For some particular cases, small (or minimal) resolutions can be directly constructed.

For instance, let $G = C_m$ with generator t. The resolution F_*

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

produces

$$H_i(G) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/m\mathbb{Z} & \text{if } i \text{ is odd} \\ 0 & \text{if } i \text{ is even}, i > 0 \end{cases}$$

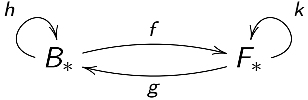
Algorithm computing the effective homology of a group

Given G a group, F_* a (small) free $\mathbb{Z}G$-resolution with a *contracting homotopy* $h_n : F_n \rightarrow F_{n+1}$.

Goal: an equivalence $C_*(K(G,1)) \Longleftrightarrow E_*$ where E_* is an effective chain complex.

We consider the bar resolution $B_* = \text{Bar}_*(G)$ for G with contracting homotopy h'.

It is well known that there exists a morphism of chain complexes of $\mathbb{Z}G$-modules $f : B_* \rightarrow F_*$ which is a homotopy equivalence. An algorithm has been designed constructing the explicit expressions of f and the corresponding maps g, h and k



Algorithm computing the effective homology of a group

Applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ we obtain an equivalence of chain complexes (of $\mathbb{Z}$-modules):

$$h \circlearrowright \mathbb{Z} \otimes_{\mathbb{Z}G} B_* \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{Z} \otimes_{\mathbb{Z}G} F_* \circlearrowleft k$$

In order to obtain a strong chain equivalence we make use of the mapping cylinder construction.

$$\mathbb{Z} \otimes_{\mathbb{Z}G} B_* \overset{\rho'}{\Lleftarrow} \text{Cylinder}(f)_* \overset{\rho}{\Rrightarrow} \mathbb{Z} \otimes_{\mathbb{Z}G} F_*$$

Finally we observe that the left chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} B_*$ is equal to $C_*(K(G,1))$. Moreover, if the initial resolution F_* is of finite type (and small), then the right chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G} F_* \equiv E_*$ is effective.

Algorithm computing the effective homology of a group

Theorem

A general algorithm can be produced:

- *Input: a group G and a free resolution F_* of finite type with contracting homotopy.*
- *Output: the effective homology of $K(G,1)$, that is, a (strong chain) equivalence $C_*(K(G,1)) \Lleftarrow\!\!\Rrightarrow E_*$ where E_* is an effective chain complex.*

- Implemented in Common Lisp, enhancing the Kenzo system.
- It allows to compute homology of groups and, what is more important, to use the space $K(G,1)$ in other constructions allowing new computations.

Zigzag persistence

- The theory of zigzag persistence provides an extension of persistent homology to diagrams of topological spaces of the form:

$$X_1 \leftrightarrow X_2 \leftrightarrow \cdots \leftrightarrow X_m$$

where the arrows can point either left or right.

- For each $n \in \mathbb{N}$, the associated sequence of vector spaces and linear maps:

$$H_n(X_1) \leftrightarrow H_n(X_2) \leftrightarrow \cdots \leftrightarrow H_n(X_m)$$

is called a *zigzag module*.

Zigzag persistence

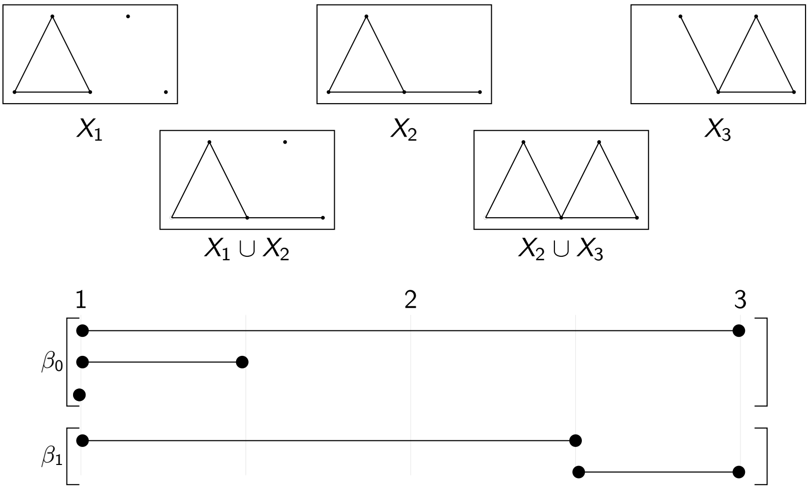
- Zigzag modules can be decomposed as a direct sum of submodules W^i of the form

$$0 \leftrightarrow \cdots \leftrightarrow 0 \leftrightarrow W^i_{a_i} = \mathbb{F} \leftrightarrow \cdots \leftrightarrow W^i_{b_i} = \mathbb{F} \leftrightarrow 0 \leftrightarrow \cdots \leftrightarrow 0$$

  for some $1 \leq a_i \leq b_i \leq m$, where $\mathbb{F}$ is the base field and all arrows are the identity map. In this way, zigzag modules can be classified up to isomorphism by a multi-set of intervals $\{[a_i, b_i]\}$ with $1 \leq a_i \leq b_i \leq m$ and represented by means of *barcode diagrams*.
- Zigzag persistence can be useful for studying the relations of the homology classes of different subspaces $X_1, \ldots, X_m$ of a topological space X when a filtration is not defined. To this aim, one considers the sequence:

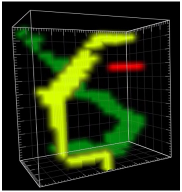
$$X_1 \hookrightarrow X_1 \cup X_2 \hookleftarrow X_2 \hookrightarrow X_2 \cup X_3 \hookleftarrow \cdots \hookrightarrow X_{m-1} \cup X_m \hookleftarrow X_m$$

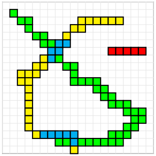
Zigzag persistence



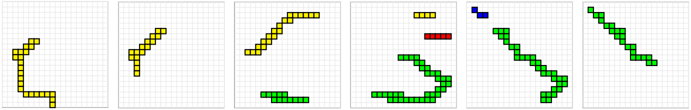
Both persistent homology and zigzag persistence allow us to detect the structure of a neuron from a stack of images.

NeuronZigZagJ

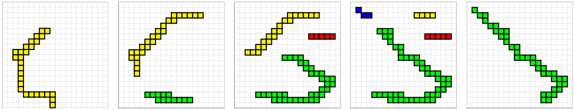




S_i



$S_i \cup S_{i+1}$



NeuronZigZagJ

Algorithm formalized by means of zigzag persistence:

- For each slice S_i we consider the associated simplicial complex, denoted X_i. It is a topological space and its homology groups in dimension 0 determine the connected components of the image S_i.
- Similarly for the simplicial complex associated to the union $S_i \cup S_{i+1}$ which is in fact equal to $X_i \cup X_{i+1}$.
- Then one has the following diagram

$$X_1 \hookrightarrow X_1 \cup X_2 \hookleftarrow X_2 \hookrightarrow X_2 \cup X_3 \hookleftarrow \cdots \hookrightarrow X_{m-1} \cup X_m \hookleftarrow X_m$$

and the corresponding zigzag module for degree 0:

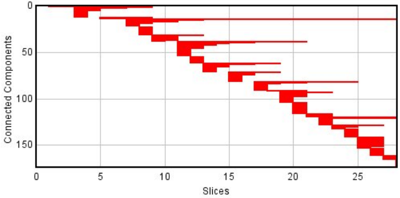
$$H_0(X_1) \to H_0(X_1 \cup X_2) \leftarrow H_0(X_2) \to H_0(X_2 \cup X_3) \leftarrow \cdots \to H_0(X_{m-1} \cup X_m) \leftarrow H_0(X_m)$$

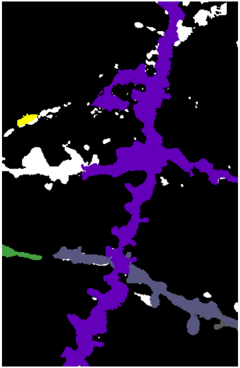
NeuronZigZagJ

- In a more realistic situation the complete 3D image is not available because the microscope provides only a stack of several 2D images $I_1, \ldots, I_m$.
- We binarize each slice and determine the maximal projection of the resulting binary images $S_1, \ldots, S_m$.
- We can apply our algorithm as in the "good" situation.
- In some cases, depending on the type of images to be studied, we replace the union $S_i \cup S_{i+1}$ by the binarization of the maximal projection of the initial images I_i and I_{i+1}.
- The algorithm returns an *approximation* of the desired projection of the different connected components of the 3D object.

NeuronZigZagJ

Output:





Conclusions

- The Basic Perturbation Lemma is not *basic*.
- Combined with the effective homology method, it can be used for computing homology and homotopy groups of different spaces and other constructions of Algebraic Topology such as spectral sequences, persistent homology, homology of groups. . .

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